

A GENERALIZED KOSZUL THEORY AND ITS RELATION TO THE CLASSICAL THEORY

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ABSTRACT. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded locally finite k -algebra such that A_0 is an arbitrary finite-dimensional algebra satisfying a certain splitting condition. In this paper we develop a generalized Koszul theory preserving many classical results. Moreover, we define a quotient graded algebra $\bar{A} = \bigoplus_{i \geq 0} \bar{A}_i$ and show that A is a generalized Koszul algebra if and only if \bar{A} is a classical Koszul algebra. We also describe an application of this theory to the extension algebras of standard modules of standardly stratified algebras.

1. INTRODUCTION

The classical Koszul theory plays an important role in the representation theory of graded algebras. However, there are a lot of structures (algebras, categories, etc) having natural gradings with non-semisimple degree 0 parts, to which the classical theory cannot be applied. Particular examples of such structures include tensor algebras generated by non-semisimple algebras A_0 and (A_0, A_0) -bimodules A_1 , and extension algebras of standard modules of standardly stratified algebras (see [13]). Therefore, we are motivated to develop a generalized Koszul theory which can be used to study above structures, and preserves many classical results such as the Koszul duality. Moreover, we also hope to get a close relation between this generalized theory and the classical theory.

In [10, 15, 16, 26] several generalized Koszul theories have been described, where the degree 0 part A_0 of a graded algebra A is not required to be semisimple. In [26], A is supposed to be both a left projective A_0 -module and a right projective A_0 -module. In Madsen's paper [16], A_0 is supposed to have finite global dimension. These requirements are too strong for us. The theory developed by Green, Reiten and Solberg in [10] works in a very general framework. The author has already developed a generalized Koszul theory in [12] under the assumption that A_0 is self-injective, and applied it to the representation theory of some categories with nice structures. In all these papers, relations between the generalized theories and the classical theory are lacking.

In this paper we loose the assumption that A_0 is self-injective (as required in [12]) and replace it by some splitting property. Explicitly, let $A = \bigoplus_{i \geq 0} A_i$ be a graded *locally finite* k -algebra generated in degrees 0 and 1, i.e., $\dim_k A_i < \infty$ and $A_1 \cdot A_i = A_{i+1}$ for all $i \geq 0$. We then define *generalized Koszul modules* and *Koszul algebras* by linear projective resolutions. If A satisfies the following splitting condition, we show that many classical results can be preserved.

(S): Every exact sequence $0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0$ of left (resp., right) A_0 -modules splits if P and Q are left (resp., right) projective A_0 -modules.

It is well known that in the classical theory *linear modules* (defined by linear projective resolutions) and *Koszul modules* (defined by a certain extension property) coincide. We have a similar result:

Theorem 1.1. *Let $A = \bigoplus_{i \geq 0} A_i$ be a locally finite graded algebra satisfying the splitting property (S). If A is a projective A_0 -module, then a graded module M is generalized Koszul if and only if it is a projective A_0 -module and the graded $\Gamma = \text{Ext}_A^*(A_0, A_0)$ -module $\text{Ext}_A^*(M, A_0)$ is generated in degree 0, i.e.,*

$$\text{Ext}_A^1(A_0, A_0) \cdot \text{Ext}_A^i(M, A_0) = \text{Ext}_A^{i+1}(M, A_0).$$

We also have the generalized Koszul duality as follows:

Theorem 1.2. *Let $A = \bigoplus_{i \geq 1} A_i$ be a locally finite graded algebra satisfying the splitting condition (S). If A is a generalized Koszul algebra, then $E = \text{Ext}_A^*(-, A_0)$ gives a duality between the category of generalized Koszul A -modules and the category of generalized Koszul $\Gamma = \text{Ext}_A^*(A_0, A_0)$ -modules. That is, if M is a Koszul A -module, then $E(M)$ is a Koszul Γ -module, and $E_\Gamma EM = \text{Ext}_\Gamma^*(EM, \Gamma_0) \cong M$.*

Let \mathfrak{r} be the radical of A_0 and define $\mathfrak{R} = A\mathfrak{r}A$ to be the two-sided ideal generated by \mathfrak{r} . For a graded A -module $M = \bigoplus_{i \geq 0} M_i$, we then define a quotient algebra $\bar{A} = A/A\mathfrak{r}A = \bigoplus_{i \geq 0} A_i/(\mathfrak{r}A)_i$ and $\bar{M} = M/\mathfrak{R}M = \bigoplus_{i \geq 0} M_i/(\mathfrak{R}M)_i$. We prove that \bar{M} is a well defined \bar{A} -module, and show that M is generated in degree 0 if and only if the corresponding graded \bar{A} -module \bar{M} is generated in degree 0. Consequently, we get the following correspondence between our generalized Koszul theory and the classical theory:

Theorem 1.3. *Let $A = \bigoplus_{i \geq 1} A_i$ be a locally finite graded algebra and M be a graded A -module. Suppose that both A and M are projective A_0 -modules. Then M is generalized Koszul if and only if the corresponding graded \bar{A} -module \bar{M} is classical Koszul. In particular, A is a generalized Koszul algebra if and only if \bar{A} is a classical Koszul algebra.*

Note that in the above theorem we do not assume the splitting condition. Therefore, it can be applied to any generalized Koszul algebras. Particular examples includes extension algebras $\Gamma = \text{Ext}_A^*(\Delta, \Delta)$ of standard modules of standardly stratified algebras A . In [13] we have described a sufficient condition for these extension algebras to be generalized Koszul. In this paper we show that their corresponding quotient algebras $\bar{\Gamma}$ are classical Koszul.

The paper is organized as follows: In the next section we develop the generalized Koszul theory and prove the first two theorems. In Section 3 we describe the relation between these two Koszul theories and prove the third theorem. We apply the correspondence to extension algebras of standard modules in the last section.

Throughout this paper k is supposed to be algebraically closed, and all modules are finitely generated (in the non-graded situation) or locally finite (in the graded situation) left modules if they are not specified. Composition of morphisms and maps are from right to left.

2. A GENERALIZED KOSZUL THEORY

We start with some preliminary results, most of which are generalized from those described in [4, 8, 9, 18, 21]. The reader is also suggested to look at other generalized Koszul theories described in [10, 15, 16, 26].

Throughout this section let $A = \bigoplus_{i \geq 0} A_i$ be a *locally finite* graded algebra generated in degrees 0 and 1, i.e., $\dim_k A_i < \infty$ and $A_{i+1} = A_1 \cdot A_i$ for all $i \geq 0$. An A -module $M = \bigoplus_{i \geq 0} M_i$ is *graded* if $A_i \cdot M_j \subseteq M_{i+j}$. It is said to be *generated in degree s* if $M = A \cdot M_s$. It is *locally finite* if $\dim_k M_i < \infty$ for all $i \geq 0$. In this paper all graded modules are supposed to be locally finite. The degree shift functor $[-]$ is defined by letting $M[i]_s = M_{s-i}$, $i, s \in \mathbb{Z}$. Denote $\mathfrak{J} = \bigoplus_{i \geq 1} A_i$, which is a two-sided ideal of A . We identify A_0 with the quotient module A/\mathfrak{J} and view it as a graded A -module concentrated in degree 0.

The following lemmas are proved in [12], where we did not use the condition that A_0 is self-injective (Remark 2.8 in [12]).

Lemma 2.1. (Lemma 2.1 in [12]) *Let A be as above and M be a graded A -module. Then:*

- (1) \mathfrak{J} is contained in the graded radical of A ;
- (2) M has a graded projective cover;
- (3) the graded syzygy ΩM is also locally finite.

Lemma 2.2. (Lemma 2.2 in [12]) *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of graded A -modules. Then:*

- (1) If M is generated in degree s , so is N .
- (2) If L and N are generated in degree s , so is M .
- (3) If M is generated in degree s , then L is generated in degree s if and only if $\mathfrak{J}M \cap L = \mathfrak{J}L$.

Now we define *generalized Koszul modules* and *generalized Koszul algebras*.

Definition 2.3. *A graded A -module M is called a generalized Koszul module if it has a (minimal) linear projective resolution*

$$\cdots \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

such that P^i is generated in degree i for all $i \geq 0$. The graded algebra A is called a generalized Koszul algebra if A_0 viewed as an A -module is Koszul.

The reader can easily see that M is a Koszul A -module if and only if M is generated in degree 0 and $\Omega^i(M)$ is generated in degree i for every $i \geq 1$. Moreover, from the above projective resolution, we deduce that $M_0 \cong P_0^0$ and $\Omega^i(M)_i \cong P_i^i$ are projective A_0 -modules for all $i \geq 1$.

From now on we suppose that A satisfies the splitting condition (S). That is, every exact sequence $0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0$ of left (resp., right) A_0 -modules splits if P and Q are left (resp., right) projective A_0 -modules.

Proposition 2.4. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of graded A -modules such that L is Koszul. Then M is generalized Koszul if and only if N is generalized Koszul.*

Proof. This is Proposition 2.9 in [12]. The proof is almost the same except replacing the self-injective property of A_0 by the splitting property (S). For the sake of completeness we give a brief proof here.

By the second statement of the previous lemma, M is generated in degree 0 if and only if N is generated in degree 0. Consider the following diagram in which all

rows and columns are exact:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega L & \longrightarrow & M' & \longrightarrow & \Omega N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P & \longrightarrow & P \oplus Q & \longrightarrow & Q \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0.
\end{array}$$

Here P and Q are graded projective covers of L and N respectively. We claim $M' \cong \Omega M$. Indeed, the given exact sequence induces an exact sequence of A_0 -modules:

$$0 \longrightarrow L_0 \longrightarrow M_0 \longrightarrow N_0 \longrightarrow 0.$$

Observe that L_0 is a projective A_0 -module. If N is Koszul, then N_0 is a projective A_0 -module since $N_0 \cong Q_0^0$, and the above sequence splits. If M is generalized Koszul, then M_0 is a projective A_0 -module, and this sequence splits as well by the splitting property (S). In either case we have $M_0 \cong L_0 \oplus N_0$. Thus $P \oplus Q$ is a graded projective cover of M , and hence $M' \cong \Omega M$ is generated in degree 1 if and only if ΩN is generated in degree 1 by Lemma 2.2. Replace L , M and N by $(\Omega L)[-1]$, $(\Omega M)[-1]$ and $(\Omega N)[-1]$ (all of them are generalized Koszul) respectively in the short exact sequence. Repeating the above procedure we prove the conclusion by recursion. \square

If M is a generalized Koszul module, its truncations (with suitable degree shifts) are generalized Koszul as well:

Proposition 2.5. *Let A be a generalized Koszul algebra and M be a generalized Koszul module. Then $\mathfrak{J}^i M[-i]$ is also generalized Koszul for each $i \geq 1$.*

Proof. This is Proposition 2.13 in [12]. For the convenience of the reader we include a brief proof here. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega M & \longrightarrow & \Omega(M_0) & \longrightarrow & \mathfrak{J}M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^0 & \xrightarrow{id} & P^0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{J}M & \longrightarrow & M & \longrightarrow & M_0 \longrightarrow 0
\end{array}$$

Since M_0 is a projective A_0 -module and A_0 is generalized Koszul, $\Omega(M_0)[-1]$ is also generalized Koszul. Similarly, $\Omega M[-1]$ is generalized Koszul since so is M . Therefore, $\mathfrak{J}M[-1]$ is generalized Koszul by the previous proposition. Now replacing M

by $\mathfrak{J}M[-1]$ and using recursion, we conclude that $\mathfrak{J}^i M[-i]$ is a generalized Koszul A -module for every $i \geq 1$. \square

From this proposition we immediately deduce that if A is a generalized Koszul algebra, then it is a projective A_0 -module. We now focus on graded algebras with this property.

Proposition 2.6. *If A is a projective A_0 -module, then every generalized Koszul module M is a projective A_0 -module.*

Proof. Clearly, it suffices to show that M_i is a projective A_0 -module for each $i \geq 0$. Since M is generalized Koszul, M_0 is a projective A_0 -module. Now suppose $i \geq 1$. The minimal linear projective resolution of M gives rise to exact sequences of A_0 -modules:

$$0 \longrightarrow \Omega^{s+1}(M)_i \longrightarrow P_i^s \longrightarrow \Omega^s(M)_i \longrightarrow 0, \quad 0 \leq s \leq i.$$

If $s = i$, we have $\Omega^{i+1}(M)_i = 0$ since $\Omega^{i+1}(M)$ is generated in degree $i+1$. Thus $\Omega^i(M)_i \cong P_i^i$ is a projective A_0 -module. Now let $s = i-1$. We claim that the first term $\Omega^i(M)_i$ is a projective A_0 -module. Indeed, $\Omega^i(M)[-i]$ is a generalized Koszul module, so $(\Omega^i(M)[-i])_0$ is a projective A_0 -module. But $\Omega^i(M)_i \cong (\Omega^i(M)[-i])_0$. This proves the claim. Since the first two terms are projective A_0 -modules, by the splitting property (S), we deduce that $\Omega^{i-1}(M)_i$ is a projective A_0 -module. By recursion, we conclude that M_i is a projective A_0 -module for every $i > 0$. \square

The following lemma will be used in the proof of Theorem 1.1.

Lemma 2.7. *Let M be a graded A -module generated in degree 0. Suppose that both A and M are projective A_0 -modules. Then ΩM is generated in degree 1 if and only if every A -module homomorphism $\Omega M \rightarrow A_0$ extends to an A -module homomorphism $\mathfrak{J}P \rightarrow A_0$, where P is a graded projective cover of M .*

Proof. This is a varied version of Lemma 2.17 in [12]. The exact sequence $0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$ induces an exact sequence $0 \rightarrow (\Omega M)_1 \rightarrow P_1 \rightarrow M_1 \rightarrow 0$ of A_0 -modules, which splits since M_1 is a projective A_0 -module. Applying the functor $\text{Hom}_{A_0}(-, A_0)$ we get another splitting exact sequence

$$0 \rightarrow \text{Hom}_{A_0}(M_1, A_0) \rightarrow \text{Hom}_{A_0}(P_1, A_0) \rightarrow \text{Hom}_{A_0}((\Omega M)_1, A_0) \rightarrow 0.$$

Note that $(\Omega M)_0 = 0$. Therefore, ΩM is generated in degree 1 if and only if $\Omega M / \mathfrak{J}(\Omega M) \cong (\Omega M)_1$, if and only if the above sequence is isomorphic to

$$0 \rightarrow \text{Hom}_{A_0}(M_1, A_0) \rightarrow \text{Hom}_{A_0}(P_1, A_0) \rightarrow \text{Hom}_{A_0}(\Omega M / \mathfrak{J}\Omega M, A_0) \rightarrow 0.$$

Here we use the fact that M_1 , P_1 and $(\Omega M)_1$ are projective A_0 -modules. But the above sequence is isomorphic to

$$0 \rightarrow \text{Hom}_A(\mathfrak{J}M, A_0) \rightarrow \text{Hom}_A(\mathfrak{J}P, A_0) \rightarrow \text{Hom}_A(\Omega M, A_0) \rightarrow 0$$

since $\mathfrak{J}M$ and $\mathfrak{J}P$ are generated in degree 1. Therefore, ΩM is generated in degree 1 if and only if every (non-graded) A -module homomorphism $\Omega M \rightarrow A_0$ extends to a (non-graded) A -module homomorphism $\mathfrak{J}P \rightarrow A_0$. \square

Let M be a graded A -module and $\Gamma = \text{Ext}_A^*(A_0, A_0)$. Then $\text{Ext}_A^*(M, A_0)$ is a graded Γ -module. Now we restate and prove Theorem 1.1.

Theorem 2.8. *Let $A = \bigoplus_{i \geq 0} A_i$ be a locally finite graded algebra satisfying the splitting property (S). If A is a projective A_0 -module, then a graded module M is generalized Koszul if and only if it is a projective A_0 -module and the graded $\Gamma = \text{Ext}_A^*(A_0, A_0)$ -module $\text{Ext}_A^*(M, A_0)$ is generated in degree 0, i.e.,*

$$\text{Ext}_A^1(A_0, A_0) \cdot \text{Ext}_A^i(M, A_0) = \text{Ext}_A^{i+1}(M, A_0).$$

Proof. This is a varied version of Theorem 2.16 in [12]. Since the proof is almost the same, we only give a sketch. Please refer to [12] for details.

The only if part. Let M be a generalized Koszul A -module. Without loss of generality we can suppose that M is indecomposable. By Lemma 2.6 M is a projective A_0 -module. As in the original proof, it suffices to show that the given identity is true for $i = 1$, i.e.,

$$\text{Ext}_A^1(M, A_0) = \text{Ext}_A^1(A_0, A_0) \cdot \text{Hom}_A(M, A_0).$$

The proof of this identity is completely the same as the original proof. We omit the details.

The if part. As in the original proof, we only need to show that ΩM is generated in degree 1. By the previous lemma, it suffices to show that each (non-graded) A -module homomorphism $g : \Omega M \rightarrow A_0$ extends to $\mathfrak{J}P^0$, where P^0 is a graded projective cover of M . The proof of this fact is completely the same as the original proof. \square

An immediate corollary of the above theorem is:

Corollary 2.9. *The graded algebra A is generalized Koszul if and only if A is a projective A_0 -module and $\Gamma = \text{Ext}_A^*(A_0, A_0)$ is generated in degrees 0 and 1.*

Now we can prove a generalized Koszul duality.

Theorem 2.10. *Let $A = \bigoplus_{i \geq 1} A_i$ be a locally finite graded algebra satisfying the splitting condition (S). If A is a generalized Koszul algebra, then $E = \text{Ext}_A^*(-, A_0)$ gives a duality between the category of generalized Koszul A -modules and the category of generalized Koszul $\Gamma = \text{Ext}_A^*(A_0, A_0)$ -modules. That is, if M is a Koszul A -module, then $E(M)$ is a Koszul Γ -module, and $E_\Gamma EM = \text{Ext}_\Gamma^*(EM, \Gamma_0) \cong M$ as graded A -modules.*

Proof. This is a varied version of Theorem 4.1 in [12]. We give a detailed proof for the convenience of the reader. Since A_0 is a generalized Koszul module and M is a projective A_0 -module, M_0 is generalized Koszul as well. By Proposition 2.5, $\mathfrak{J}M[-1]$ is also generalized Koszul. Furthermore, we have the following short exact sequence of generalized Koszul modules:

$$0 \longrightarrow \Omega M[-1] \longrightarrow \Omega(M_0)[-1] \longrightarrow \mathfrak{J}M[-1] \longrightarrow 0.$$

As in the proof of Proposition 2.4, this sequence induces exact sequences of generalized Koszul modules recursively:

$$0 \longrightarrow \Omega^i(M)[-i] \longrightarrow \Omega^i(M_0)[-i] \longrightarrow \Omega^{i-1}(\mathfrak{J}M[-1])[1-i] \longrightarrow 0.$$

Take a fixed sequence for a certain $i > 0$. It gives a splitting exact sequence of A_0 -modules:

$$0 \longrightarrow \Omega^i(M)_i \longrightarrow \Omega^i(M_0)_i \longrightarrow \Omega^{i-1}(\mathfrak{J}M[-1])_{i-1} \longrightarrow 0.$$

Applying $\text{Hom}_{A_0}(-, A_0)$ to it and using the following isomorphism for a graded A -module N generated in degree i

$$\text{Hom}_A(N, A_0) \cong \text{Hom}_A(N_i, A_0) \cong \text{Hom}_{A_0}(N_i, A_0),$$

we get:

$$0 \rightarrow \text{Hom}_A(\Omega^{i-1}(\mathfrak{J}M[-1]), A_0) \rightarrow \text{Hom}_A(\Omega^i(M_0), A_0) \rightarrow \text{Hom}_A(\Omega^i M, A_0) \rightarrow 0,$$

which is isomorphic to

$$0 \rightarrow \text{Ext}_A^{i-1}(\mathfrak{J}M[-1], A_0) \rightarrow \text{Ext}_A^i(M_0, A_0) \rightarrow \text{Ext}_A^i(M, A_0) \rightarrow 0.$$

Now let the index i vary and put these sequences together. We have:

$$0 \longrightarrow E(\mathfrak{J}M[-1])[1] \longrightarrow E(M_0) \longrightarrow EM \longrightarrow 0.$$

Let us focus on this sequence. We claim $\Omega(EM) \cong E(\mathfrak{J}M[-1])[1]$. Indeed, since M_0 is a projective A_0 -module, $E(M_0)$ is a projective Γ -module. But $\mathfrak{J}M[-1]$ is generalized Koszul, so $E(\mathfrak{J}M[-1])$ is generated in degree 0 by the previous theorem. Thus $E(\mathfrak{J}M[-1])[1]$ is generated in degree 1, and $E(M_0)$ is a graded projective cover of EM . This proves the claim. Consequently, $\Omega(EM)$ is generated in degree 1. Moreover, replacing M by $\mathfrak{J}M[-1]$ (which is also generalized Koszul) and using the claimed identity, we have that

$$\Omega^2(EM) = \Omega(E(\mathfrak{J}M[-1])[1]) = \Omega(E(\mathfrak{J}M[-1])[1]) = E(\mathfrak{J}^2 M[-2])[2],$$

is generated in degree 2. By recursion, $\Omega^i(EM) \cong E(\mathfrak{J}^i M[-i])[i]$ is generated in degree i for all $i \geq 0$. Thus EM is a generalized Koszul Γ -module (note that $\Gamma_0 \cong A_0^{\text{op}}$ also satisfies the splitting property (S)!). In particular for $M = {}_A A$,

$$EA = \text{Ext}_A^*(A, A_0) = \text{Hom}_A(A, A_0) = \Gamma_0$$

is a generalized Koszul Γ -module.

Since $\Omega^i(EM)$ is generated in degree i ,

$$\begin{aligned} \Omega^i(EM)_i &\cong E(\mathfrak{J}^i M[-i])[i]_i \cong E(\mathfrak{J}^i M[-i])_0 \\ &= \text{Hom}_A(\mathfrak{J}^i M[-i], A_0) \cong \text{Hom}_A(M_i, A_0). \end{aligned}$$

We also have

$$\begin{aligned} \text{Hom}_\Gamma(\Omega^i(EM), \Gamma_0) &\cong \text{Hom}_{\Gamma_0}(\Omega^i(EM)_i, \Gamma_0) \\ &\cong \text{Hom}_{\Gamma_0}(\text{Hom}_A(M_i, A_0), \Gamma_0) \\ &\cong \text{Hom}_{\Gamma_0}(\text{Hom}_{A_0}(M_i, A_0), \Gamma_0) \\ &\cong M_i. \end{aligned}$$

The last isomorphism holds because M_i is a projective A_0 -module and $\Gamma_0 \cong A_0^{\text{op}}$. Therefore, we get

$$\text{Ext}_\Gamma^i(EM, \Gamma_0) \cong \text{Hom}_\Gamma(\Omega^i(EM), \Gamma_0) \cong M_i$$

for every $i \geq 0$. Adding them together, $E_\Gamma E(M) \cong \bigoplus_{i=0}^\infty M_i \cong M$.

Now we have $E_\Gamma(E(A)) = E_\Gamma(\Gamma_0) \cong A$. Moreover, Γ is a graded algebra such that $\Gamma_0 \cong A_0^{\text{op}}$ is self-injective as an algebra and Koszul as a Γ -module. Using this duality, we can exchange A and Γ in the above reasoning and get $EE_\Gamma(N) \cong N$ for an arbitrary Koszul Γ -module N . Thus E is a dense functor.

Let L be another Koszul A -module. Since L, M, EL, EM are all generated in degree 0, we have

$$\begin{aligned} \mathrm{hom}_\Gamma(EL, EM) &\cong \mathrm{Hom}_{\Gamma_0}((EL)_0, (EM)_0) \\ &\cong \mathrm{Hom}_{\Gamma_0}(\mathrm{Hom}_A(L, A_0), \mathrm{Hom}_A(M, A_0)) \\ &\cong \mathrm{Hom}_{A_0^{\mathrm{op}}}(\mathrm{Hom}_{A_0}(L_0, A_0), \mathrm{Hom}_{A_0}(M_0, A_0)) \\ &\cong \mathrm{Hom}_{A_0}(L_0, M_0) \cong \mathrm{hom}_A(L, M). \end{aligned}$$

Consequently, E is a duality between the category of Koszul A -modules and the category of Koszul Γ -modules. \square

3. A RELATION BETWEEN THE GENERALIZED THEORY AND THE CLASSICAL THEORY

In [12] we have already described a correspondence between the generalized Koszul theory and the classical theory for *directed categories* \mathcal{A} , which are k -linear categories such that there are partial orders \leq on $\mathrm{Ob} \mathcal{A}$ satisfying the condition that $x \leq y$ whenever $\mathcal{A}(x, y) \neq 0$. We first collect some results on directed categories and briefly describe this correspondence. In this paper all directed categories \mathcal{A} are supposed to satisfy the following conditions: \mathcal{A} is skeletal and has only finitely many objects; \mathcal{A} is *locally finite*, i.e., $\dim_k \mathcal{A}(x, y) < \infty$ for all $x, y \in \mathrm{Ob} \mathcal{A}$; the endomorphism algebra of every object is a local algebra. We also suppose that \mathcal{A} is graded such that $\mathcal{A}_0 = \bigoplus_{x \in \mathrm{Ob} \mathcal{A}} \mathcal{A}(x, x)$. We then define

$$\mathfrak{J} = \bigoplus_{\substack{x \neq y; \\ x, y \in \mathrm{Ob} \mathcal{A}}} \mathcal{A}(x, y),$$

which coincides with $\bigoplus_{i \geq 1} \mathcal{A}_i$. Note that \mathcal{A} can be viewed as a graded algebra.

We collect some basic properties of directed categories in the following propositions.

Proposition 3.1. *Let \mathcal{A} be a directed category with respect to a partial order \leq . Then the following are equivalent:*

- (1) \mathcal{A} is standardly stratified for \leq ;
- (2) for each pair of objects $x, y \in \mathrm{Ob} \mathcal{A}$, $\mathcal{A}(x, y)$ is a free $\mathcal{A}(y, y)$ -module;
- (3) \mathcal{A} is a left projective \mathcal{A}_0 -module.

Proof. (1) \Rightarrow (2): This is a part of Theorem 5.7 in [12].

(2) \Rightarrow (3): Since $\mathcal{A}_0 = \bigoplus_{x \in \mathrm{Ob} \mathcal{A}} \mathcal{A}(x, x)$, for $x, y \in \mathrm{Ob} \mathcal{A}$, $\mathcal{A}(x, y)$ is a projective \mathcal{A}_0 module. But as an \mathcal{A}_0 -module, $\mathcal{A} \cong \bigoplus_{x, y \in \mathrm{Ob} \mathcal{A}} \mathcal{A}(x, y)$.

(3) \Rightarrow (1): By proposition 5.5 in [12], standard modules are precisely indecomposable summands of \mathcal{A}_0 (viewed as an \mathcal{A} -module) up to isomorphism. If \mathcal{A} is a projective \mathcal{A}_0 -module, then it has a filtration by standard modules, so \mathcal{A} is standardly stratified for \leq . \square

Gabriel's construction relates a finite-dimensional algebra A and a locally finite k -linear category \mathcal{A} with finitely many objects. Call \mathcal{A} the *associated category* of A and call A the *associated algebra* of \mathcal{A} (see [13] for more details) and. In the next proposition we describe some algebras whose associated categories are directed.

Proposition 3.2. *Let A be a basic finite-dimensional algebra.*

- (1) If A is a standardly stratified for a preorder \preceq , then the associated category \mathcal{E} of $E = \text{Ext}_A^*(\Delta, \Delta)$ is a directed category, where Δ is the direct sum of all standard modules.
- (2) If A is standardly stratified for all preorders, then the associated category \mathcal{A} is a directed category.

Proof. The first statement is a part of Theorem 1.4 in [13]. The second statement follows from Proposition 1.2 in [14]. \square

When applying the generalized Koszul theory to graded directed categories, we get some close relation between this generalized theory and the classical theory. The explicit correspondence is described as follows. Let \mathcal{A} be a graded directed category with respect to a partial order \leq . We define \mathcal{B} to be the graded subcategory of \mathcal{A} formed by replacing the endomorphism algebra of every object by $k \cdot 1$, the span of the identity endomorphism. That is, $\text{Ob } \mathcal{B} = \text{Ob } \mathcal{A}$; $\mathcal{B}(x, y) = \mathcal{A}(x, y)$ if $x \neq y$ and $\mathcal{B}(x, x) = k \cdot 1_x$. Let A and B be the associated graded algebras of \mathcal{A} and \mathcal{B} respectively. Then we have $\bigoplus_{i \geq 1} A_i = \bigoplus_{i \geq 1} B_i$. Note that B_0 is a semisimple algebra, so the classical theory can be applied. On the other hand, A_0 as a direct sum of several finite-dimensional local algebras satisfies the splitting property (S), so we can use the generalized Koszul theory developed in the last section.

Theorem 3.3. *Let A and B be defined as above.*

- (1) Suppose that A is a generalized Koszul algebra. If M is a generalized Koszul A -module, then the restricted module $M \downarrow_B^A$ is classical Koszul. In particular, B is a classical Koszul algebra.
- (2) Suppose that B is a classical Koszul algebra. If M is a graded A -module satisfying that $\Omega^i(M)_i$ is a projective A_0 -module for each $i \geq 0$ and $M \downarrow_B^A$ is classical Koszul, then M is generalized Koszul.

Proof. These two statements are precisely Theorems 5.13 and 5.14 in [12]. In the original proofs we did not assume that A_0 is self-injective, see Remark 5.15. \square

Theorem 3.4. *Let A and B be defined as above.*

- (1) A is a generalized Koszul algebra if and only if it is a projective A_0 -module and B is a classical Koszul algebra.
- (2) Suppose that A is a generalized Koszul algebra. Then a graded A -module M is generalized Koszul if and only if it is a projective A_0 -module and $M \downarrow_B^A$ is classical Koszul.

Proof. This is Theorem 5.16 in [12], but we drop the unnecessary condition that A_0 is self-injective.

(1). If A is a generalized Koszul algebra, then it is a projective A_0 -module, see the paragraph after Proposition 2.5. By (1) of the previous theorem, B is a classical Koszul algebra. Conversely, if B is a classical Koszul algebra, then $A_0 \downarrow_B^A$ is a classical Koszul B -module since it is a projective B_0 -module. Thus by (2) of the previous theorem, A is a generalized Koszul algebra if we can show that $\Omega^i(A_0)_i$ is a projective A_0 -module for each $i \geq 0$. We prove a stronger statement, that is, $\Omega^i(A_0)$ is a projective A_0 -module for each $i \geq 0$.

Clearly, $\Omega^0(A_0) = A_0$ is a projective A_0 -module. Consider the exact sequence

$$0 \longrightarrow \Omega^{i+1}(A_0) \longrightarrow P^i \longrightarrow \Omega^i(A_0) \longrightarrow 0.$$

By the induction hypothesis, $\Omega^i(A_0)$ is a projective A_0 -module. Thus the above sequence splits as A_0 -modules. But P^i is a projective A_0 -module since we assume that A is a projective A_0 -module, so is $\Omega^{i+1}(A_0)$. This proves (1).

(2). Since A is a generalized Koszul algebra, it is a projective A_0 -module. If M is generalized Koszul, then it is a projective A_0 -module (Proposition 2.6) and $M \downarrow_B^A$ is classical Koszul (by (1) of Theorem 3.3). Conversely, if $M \downarrow_B^A$ is classical Koszul, to prove that M is generalized Koszul, by (2) of Theorem 3.3 it suffices to show that $\Omega^i(M)_i$ is a projective A_0 -module for every $i \geq 0$. This can be proved by a similar induction as we just did. \square

Now we describe another correspondence which works in a more general situation. As before, let $A = \bigoplus_{i \geq 0} A_i$ be a locally finite graded algebra generated in degrees 0 and 1. At this moment we do **not** need the splitting condition (S). Let \mathfrak{r} be the radical of A_0 , and $\mathfrak{R} = A\mathfrak{r}A$ be the two-sided ideal generated by \mathfrak{r} . Note that $\mathfrak{r} + \mathfrak{J}$ is also a two sided-ideal of A where $\mathfrak{J} = \bigoplus_{i \geq 1} A_i$, and it coincides with the radical of A if A is finite-dimensional. We then define the quotient graded algebra $\bar{A} = A/\mathfrak{R} = \bigoplus_{i \geq 0} A_i/\mathfrak{R}_i$.

Lemma 3.5. *Notation as above, $\mathfrak{R}_s = \sum_{i=0}^s A_i \mathfrak{r} A_{s-i} = (\mathfrak{J} + \mathfrak{r})_s^{s+1}$, and \bar{A} is a well defined graded algebra.*

Proof. Since $\mathfrak{R} = A\mathfrak{r}A$, the first identity is clear. Now we prove the second one. This is clearly true for $s = 0$ since $\sum_{i=0}^0 A_i \mathfrak{r} A_{s-i} = A_0 \mathfrak{r} A_0 = \mathfrak{r} = (\mathfrak{r} + \mathfrak{J})_0$. If $s \geq 1$, then $A_i \mathfrak{r} A_{s-i} \subseteq A_s \subseteq \mathfrak{J}^i \cdot \mathfrak{r} \cdot \mathfrak{J}^{s-i} \subseteq (\mathfrak{r} + \mathfrak{J})^{s+1}$. Therefore, $\sum_{i=0}^s A_i \mathfrak{r} A_{s-i} \subseteq (\mathfrak{r} + \mathfrak{J})^{s+1}$. But every element is homogeneous and has degree s , so $\sum_{i=0}^s A_i \mathfrak{r} A_{s-i} \subseteq (\mathfrak{r} + \mathfrak{J})_s^{s+1}$.

On the other hand,

$$(\mathfrak{r} + \mathfrak{J})^{s+1} = \sum_{\substack{X_i = \mathfrak{r}, \mathfrak{J}, \\ i=0}}^{s+1} X_1 \cdots X_{s+1}.$$

Correspondingly,

$$(\mathfrak{r} + \mathfrak{J})_s^{s+1} = \sum_{\substack{X_i = \mathfrak{r}, \mathfrak{J}, \\ i=0}}^{s+1} (X_1 \cdots X_{s+1})_s.$$

Clearly, if all $X_i = \mathfrak{J}$ for $0 \leq i \leq s+1$, then $(X_1 \cdots X_{s+1})_s = 0$. So we can assume that there is at least one $X_i = \mathfrak{r}$.

Take $0 \neq v_s \in (X_1 \cdots X_{s+1})_s$. If $X_1 = \mathfrak{r}$, then $v_s \in \mathfrak{r} A_s$; if $X_{s+1} = \mathfrak{r}$, then $v_s \in A_s \mathfrak{r}$. Otherwise, we have some $0 < t < s+1$ such that $X_i = \mathfrak{r}$, and hence $v_s \in (\mathfrak{J} \cdot \mathfrak{r} \cdot \mathfrak{J})_s = \sum_{i=1}^s A_i \mathfrak{r} A_{s-i}$. In all cases we have $v_s \in \sum_{i=0}^s A_i \mathfrak{r} A_{s-i}$. Therefore, $(\mathfrak{r} + \mathfrak{J})_s^{s+1} \subseteq \sum_{i=0}^s A_i \mathfrak{r} A_{s-i}$. This proves the first statement.

The product of \bar{A} is defined by the following rule. Take $a_s \in A_s$ and $a_t \in A_t$, $s, t \geq 0$, we define $\bar{a}_s \cdot \bar{a}_t = \overline{a_s a_t}$ to be the image of $a_s a_t$ in $\bar{A}_{s+t} = A_{s+t}/\mathfrak{R}_{s+t}$. Since by the first statement,

$$\bar{A}_s = A_s / \sum_{i=0}^s A_i \mathfrak{r} A_{s-i}, \quad \bar{A}_t = A_t / \sum_{i=0}^t A_i \mathfrak{r} A_{t-i}, \quad \bar{A}_{s+t} = A_{s+t} / \sum_{i=0}^{s+t} A_i \mathfrak{r} A_{s+t-i},$$

it is enough to show that

$$\sum_{i=0}^s A_i \mathfrak{r} A_{s-i} \cdot A_t \subseteq \sum_{i=0}^{s+t} A_i \mathfrak{r} A_{s+t-i}, \quad A_s \cdot \sum_{i=0}^t A_i \mathfrak{r} A_{t-i} \subseteq \sum_{i=0}^{s+t} A_i \mathfrak{r} A_{s+t-i}.$$

But these two inclusions hold obviously. Therefore, the product defined in this way gives rise to a well define product of \bar{A} by bilinearity. \square

Note that $\bar{A}_0 = A_0/\mathfrak{r}$ is a semisimple algebra.

Given an arbitrary graded A -module $M = \bigoplus_{i \geq 0} M_i$, we can define a graded \bar{A} -module $\bar{M} = M/\mathfrak{R}M = \bigoplus_{i \geq 0} M_i/(\mathfrak{R}M)_i$.

Lemma 3.6. *Let \bar{M} be as above. Then*

- (1) $(\mathfrak{R}M)_n = \sum_{i=0}^n A_i \mathfrak{r} M_{n-i}$;
- (2) \bar{M} is a well defined \bar{A} -module;
- (3) if M is generated in degree 0, then

$$(\mathfrak{R}M)_n = \sum_{i=0}^n A_i \mathfrak{r} M_{n-i} = \sum_{i=0}^n A_i \mathfrak{r} A_{n-i} M_0 = ((\mathfrak{r} + \mathfrak{J})^{n+1} M)_n.$$

Proof. Since $A_i \mathfrak{r} \in \mathfrak{R}$, we have $A_i \mathfrak{r} M_{n-i} \subseteq (\mathfrak{R}M)_i$. Letting i vary we get $(\mathfrak{R}M)_n \supseteq \sum_{i=0}^n A_i \mathfrak{r} M_{n-i}$. On the other hand,

$$(\mathfrak{R}M)_n = \sum_{s=0}^n \mathfrak{R}_s M_{n-s} = \sum_{s=0}^n \sum_{i=0}^s A_i \mathfrak{r} A_{s-i} M_{n-s} \subseteq \sum_{s=0}^n \sum_{i=0}^s A_i \mathfrak{r} M_{n-i} = \sum_{i=0}^n A_i \mathfrak{r} M_{n-i}.$$

This proves the first statement.

Now we prove the second statement. Take $a_s \in A_s$ and $v_t \in M_t$. We define $\bar{a}_s \cdot \bar{v}_t$ to be the image $\overline{a_s v_t}$ of $a_s v_t$ in \bar{M}_{s+t} . Since

$$\bar{A}_s = A_s / \sum_{i=0}^s A_i \mathfrak{r} A_{s-i}, \quad \bar{M}_t = M_t / \sum_{i=0}^t A_i \mathfrak{r} M_{t-i}, \quad \bar{M}_{s+t} = M_{s+t} / \sum_{i=0}^{s+t} A_i \mathfrak{r} M_{s+t-i},$$

it suffices to show the following two inclusions:

$$\sum_{i=0}^s A_i \mathfrak{r} A_{s-i} \cdot M_t \subseteq \sum_{i=0}^{s+t} A_i \mathfrak{r} M_{s+t-i}, \quad A_s \cdot \sum_{i=0}^t A_i \mathfrak{r} M_{t-i} \subseteq \sum_{i=0}^{s+t} A_i \mathfrak{r} M_{s+t-i}.$$

But these two inclusions are clearly true. Thus \bar{M} is a well defined graded \bar{A} -module.

The first identity in the third statement has been established in (1). The second identity is clearly true since M is generated in degree 0. So we only need to show the last identity. It is clear that

$$((\mathfrak{r} + \mathfrak{J})^{n+1} M)_n = \sum_{i=0}^n ((\mathfrak{r} + \mathfrak{J})^{n+1})_i M_{n-i} = \sum_{i=0}^n ((\mathfrak{r} + \mathfrak{J})^{n+1})_i A_{n-i} M_0.$$

By taking $i = n$ and using the previous lemma, we have

$$\sum_{i=0}^n A_i \mathfrak{r} A_{n-i} M_0 = ((\mathfrak{r} + \mathfrak{J})^{n+1})_n M_0 \subseteq ((\mathfrak{r} + \mathfrak{J})^{n+1} M)_n.$$

But on the other hand,

$$\begin{aligned}
\sum_{i=0}^n ((\mathfrak{r} + \mathfrak{J})^{n+1})_i A_{n-i} M_0 &\subseteq \sum_{i=0}^n ((\mathfrak{r} + \mathfrak{J})^{n+1})_i \cdot ((\mathfrak{r} + \mathfrak{J})^{n-i})_{n-i} M_0 \\
&\subseteq \sum_{i=0}^n ((\mathfrak{r} + \mathfrak{J})^{2n+1-i})_n M_0 \\
&\subseteq ((\mathfrak{r} + \mathfrak{J})^{n+1})_n M_0 \\
&= \sum_{i=0}^n A_i \mathfrak{r} A_{n-i} M_0
\end{aligned}$$

since for $0 \leq i \leq n$ we always have $(\mathfrak{r} + \mathfrak{J})^{2n+1-i} \subseteq (\mathfrak{r} + \mathfrak{J})^{n+1}$. This finishes the proof. \square

We use an example to show our construction.

Example 3.7. Let A be the path algebra of the following quiver with relations: $\delta^2 = \theta^2 = 0$, $\theta\alpha = \alpha\delta = \beta$. Put $A_0 = \langle 1_x, 1_y, \delta, \theta \rangle$ and $A_1 = \langle \alpha, \beta \rangle$.

$$\begin{array}{ccc}
\circ & \xrightleftharpoons[\beta]{\alpha} & \circ \\
\curvearrowright & & \curvearrowleft
\end{array}$$

The structures of graded indecomposable projective A -modules are:

$$P_x = \begin{array}{ccc} & x_0 & \\ & \downarrow & \\ x_0 & & y_1 \\ & \uparrow & \\ & y_1 & \end{array} \quad P_y = \begin{array}{ccc} & y_0 & \\ & \downarrow & \\ y_0 & & y_0 \\ & \uparrow & \\ & y_0 & \end{array}.$$

We find $\mathfrak{r} = \langle \delta, \theta \rangle$, $\mathfrak{R} = \langle \delta, \theta, \beta \rangle$. Then the quotient algebra \bar{A} is the path algebra of the following quiver with a natural grading:

$$\begin{array}{ccc}
\circ & \xrightarrow{\alpha} & \circ \\
\curvearrowright & & \curvearrowleft
\end{array}$$

Let $M = \text{rad } P_x = \langle \delta, \alpha, \beta \rangle$ which is a graded A -module. This module has the following structure and is not generated in degree 0:

$$M = \begin{array}{ccc} & x_0 & \\ & \downarrow & \\ & y_1 & \\ & \uparrow & \\ & y_1 & \end{array}.$$

Then $\bar{M}_0 = M_0 / \mathfrak{r} M_0 = \langle \bar{\delta} \rangle \cong \bar{S}_x$, $\bar{M}_1 = M_1 / (\mathfrak{r} M_1 + A_1 \mathfrak{r} M_0) = \langle \bar{\alpha} \rangle \cong \bar{S}_y[1]$. Therefore, $\bar{M} \cong \bar{S}_x \oplus \bar{S}_y[1]$ is a direct sum of two simple \bar{A} -modules, and is not generated in degree 0 either.

We also note that since M is not generated in degree 0, the identities in the third statement of the previous lemma are no long true. Indeed, we have:

$$\begin{aligned}
((\mathfrak{r} + \mathfrak{J})^2 M)_1 &= \sum_{i=0}^1 A_i \mathfrak{r} A_{1-i} M_0 = \mathfrak{r} A_1 M_0 + A_1 \mathfrak{r} M_0 = 0; \\
(\mathfrak{R} M)_1 &= \sum_{i=0}^1 A_i \mathfrak{r} M_{1-i} = \mathfrak{r} M_1 + A_1 \mathfrak{r} M_0 = \langle \beta \rangle.
\end{aligned}$$

The following proposition is crucial to prove the correspondence.

Proposition 3.8. A graded A -module M is generated in degree 0 if and only if the corresponding graded \bar{A} -module \bar{M} is generated in degree 0.

Proof. If M is generated in degree 0, then $A_i M_0 = M_i$ for all $i \geq 0$. By our definition, it is clear that $\bar{A}_i \bar{M}_0 = \bar{M}_i$. That is, \bar{M} is generated in degree 0.

Conversely, suppose that \bar{M} is generated in degree 0. We want to show $A_i M_0 = M_i$ for $i \geq 0$. We use induction to prove this identity. Clearly, it holds for $i = 0$. So we suppose that it is true for all $0 \leq i < n$ and consider M_n .

Take $v_n \in M_n$ and consider its image \bar{v}_n in $\bar{M}_n = M_n / \sum_{i=0}^n A_i \mathfrak{r} M_{n-i}$. Since \bar{M} is generated in degree 0, we can find some $a_n \in A_n$ and $v_0 \in M_0$ such that $\bar{v}_n = \bar{a}_n \bar{v}_0$. Thus $\overline{v_n - a_n v_0} = \bar{v}_n - \bar{a}_n \bar{v}_0 = 0$. This means

$$v_n - a_n v_0 \in \sum_{i=0}^n A_i \mathfrak{r} M_{n-i} = \mathfrak{r} M_n + \sum_{i=1}^n A_i \mathfrak{r} M_{n-i} = \mathfrak{r} M_n + \sum_{i=1}^n A_i \mathfrak{r} A_{n-i} M_0,$$

where the last identity follows from the induction hypothesis. But it is clear $A_n M_0 \supseteq \sum_{i=1}^n A_i \mathfrak{r} A_{n-i} M_0$, so $v_n - a_n v_0 \in \mathfrak{r} M_n + A_n M_0$. Consequently, $v_n \in \mathfrak{r} M_n + A_n M_0$. Since $v_n \in M_n$ is arbitrary, we have $M_n \subseteq \mathfrak{r} M_n + A_n M_0$. Applying Nakayama's lemma to these A_0 -modules, we conclude that $M_n = A_n M_0$ as well. The conclusion then follows from induction. \square

Lemma 3.9. *Let M be a graded A -module generated in degree 0. If P is a grade projective cover of M , then \bar{P} is a graded projective cover of \bar{M} .*

Proof. Clearly, \bar{P} is a grade projective module. Both \bar{P} and \bar{M} are generated in degree 0 by the previous proposition. To show that \bar{P}_0 is a graded projective cover of \bar{M}_0 , it suffices to show that \bar{P}_0 is a projective cover of \bar{M}_0 as \bar{A}_0 -modules. But this is clearly true since $\bar{P}_0 = P_0 / \mathfrak{r} P_0 \cong M_0 / \mathfrak{r} M_0 \cong \bar{M}_0$. \square

The procedure of sending M to \bar{M} preserves exact sequences of graded A -modules which are projective A_0 -modules.

Lemma 3.10. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of graded A -modules such that all terms are projective A_0 -modules. Then the corresponding sequence $0 \rightarrow \bar{L} \rightarrow \bar{M} \rightarrow \bar{N} \rightarrow 0$ is also exact.*

Proof. For each $i \geq 0$, the given exact sequence induces a short exact sequence of A_0 -modules $0 \rightarrow L_i \rightarrow M_i \rightarrow N_i \rightarrow 0$. Since all terms are projective A_0 -modules, this sequence splits. Thus we get an exact sequence of \bar{A}_0 -modules $0 \rightarrow \bar{L}_i \rightarrow \bar{M}_i \rightarrow \bar{N}_i \rightarrow 0$. Let the index i vary and take direct sum. Then we get an exact sequence of graded \bar{A} -modules $0 \rightarrow \bar{L} \rightarrow \bar{M} \rightarrow \bar{N} \rightarrow 0$ as claimed. \square

The condition that all terms are projective A_0 -modules cannot be dropped, as shown by the following example.

Example 3.11. *Let $A = A_0 = k[t]/(t^2)$ and S be the simple module and consider a short exact sequence of graded A -modules $0 \rightarrow S \rightarrow A \rightarrow S \rightarrow 0$. We have $\bar{A} \cong k$. But the corresponding sequence $0 \rightarrow \bar{S} \rightarrow \bar{A} \rightarrow \bar{S} \rightarrow 0$ is not exact. Actually, the first map $\bar{S} \rightarrow \bar{A}$ is 0 since the image of S is contained in $\mathfrak{r} A_0$.*

Now we can prove the main result of this section.

Theorem 3.12. *Let $A = \bigoplus_{i \geq 1} A_i$ be a locally finite graded algebra and M be a graded A -module. Suppose that both A and M are projective A_0 -modules. Then M is generalized Koszul if and only if the corresponding grade \bar{A} -module \bar{M} is classical Koszul. In particular, A is a generalized Koszul algebra if and only if \bar{A} is a classical Koszul algebra.*

Proof. Let

$$(3.1) \quad \dots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of M . Note that all terms in this resolution and all syzygies are projective A_0 -modules. By Lemmas 3.9 and 3.10, \bar{M} has the following minimal projective resolution

$$(3.2) \quad \dots \longrightarrow \bar{P}^2 \longrightarrow \bar{P}^1 \longrightarrow \bar{P}^0 \longrightarrow \bar{M} \longrightarrow 0.$$

Moreover, this resolution is linear if and only if the resolution (3.1) is linear by Proposition 3.8. That is, M is generalized Koszul if and only if \bar{M} is classical Koszul. This proves the first statement. Applying it to the graded A -module A_0 we deduce the second statement immediately. \square

If A has the splitting property (S), we have a corresponding version for the previous theorem.

Corollary 3.13. *Let $A = \bigoplus_{i \geq 1} A_i$ be a locally finite graded algebra satisfying the splitting property (S).*

- (1) *A is a generalized Koszul algebra if and only if it is a projective A_0 -module and \bar{A} is a classical Koszul algebra.*
- (2) *Suppose that A is a projective A_0 -module. A graded A -module M is generalized Koszul if and only if it is a projective A_0 -module and the corresponding grade \bar{A} -module \bar{M} is classical Koszul.*

Proof. If A is a generalized Koszul algebra, then applying Proposition 2.5 to ${}_A A$ we conclude that it is a projective A_0 -module. Moreover, \bar{A} is a classical Koszul algebra by the previous theorem. The converse statement also follows from the previous theorem. This proves the first statement.

If A is a projective A_0 -module and M is generalized Koszul, by Proposition 2.6 M is a projective A_0 -module. Moreover, \bar{M} is a classical Koszul module by the previous theorem. The converse statement follows from the previous theorem as well. \square

We cannot drop the condition that A is a projective A_0 -module in the above theorem, as shown by the following example.

Example 3.14. *Let A be the path algebra of the following quiver with relations: $\delta^2 = \theta^2 = 0$, $\theta\alpha = \alpha\delta = 0$. Put $A_0 = \langle 1_x, 1_y, \delta, \theta \rangle$ and $A_1 = \langle \alpha \rangle$.*

$$\begin{array}{ccc} \delta \circlearrowleft & x \xrightarrow{\alpha} & y \circlearrowright \theta \end{array}$$

The structures of graded indecomposable projective A -modules are:

$$P_x = \begin{array}{cc} & x_0 \\ x_0 & y_1 \end{array} \quad P_y = \begin{array}{cc} & y_0 \\ y_0 & y_0 \end{array}.$$

We find $\mathfrak{r} = \langle \delta, \theta \rangle = \mathfrak{R}$. Then the quotient algebra \bar{A} is the path algebra of the following quiver:

$$1_x \circlearrowleft x \xrightarrow{\alpha} y \circlearrowright 1_y.$$

Let $\Delta_x = P_x/S_y = \langle \delta, 1_x \rangle$ which is a graded A -module concentrated in degree 0. The first syzygy $\Omega(\Delta_x) \cong S_y[1]$ is generated in degree 1, but the second syzygy $\Omega^2(\Delta_x) \cong S_y[1]$ is not generated in degree 2. Therefore, Δ_x is not generalized Koszul. However, $\bar{\Delta}_x \cong \bar{S}_x$ is obviously a classical Koszul \bar{A} -module. Moreover,

we can check that A is not a generalized Koszul algebra, but \bar{A} is a classical Koszul algebra.

4. EXTENSION ALGEBRAS OF STANDARD MODULES

Let A be a basic finite-dimensional algebra whose simple modules S_λ (up to isomorphism) are indexed by a preordered set (Λ, \leq) . This preordered set also indexes all indecomposable projective A -modules P_λ up to isomorphism. According to [5], A is *standardly stratified* with respect to \leq if there exist modules Δ_λ , $\lambda \in \Lambda$, such that the following conditions hold:

- (1) the composition factor multiplicity $[\Delta_\lambda : S_\mu] = 0$ whenever $\mu \not\leq \lambda$; and
- (2) for every $\lambda \in \Lambda$ there is a short exact sequence $0 \rightarrow K_\lambda \rightarrow P_\lambda \rightarrow \Delta_\lambda \rightarrow 0$ such that K_λ has a filtration with factors Δ_μ where $\mu > \lambda$.

The *standard module* Δ_λ is the largest quotient of P_λ with only composition factors S_μ such that $\mu \leq \lambda$. It has the following description:

$$\Delta_\lambda = P_\lambda / \sum_{\mu \not\leq \lambda} \text{tr}_{P_\mu}(P_\lambda),$$

where $\text{tr}_{P_\mu}(P_\lambda)$ is the trace of P_μ in P_λ ([6, 25]).

Throughout this section we suppose that A is standardly stratified with respect to \leq . Let Δ be the direct sum of all standard modules and $\mathcal{F}(\Delta)$ be the full subcategory of $A\text{-mod}$ such that each object in $\mathcal{F}(\Delta)$ has a filtration by standard modules. For $M \in \mathcal{F}(\Delta)$ and $\lambda \in \Lambda$, we take a particular Δ -filtration ξ and define the multiplicity $[M : \Delta_\lambda]$ to be the number of factors in ξ isomorphic to Δ_λ . It is well known that the multiplicity is independent of the choice of a particular filtration.

Since standard modules are relative simple in the category $\mathcal{F}(\Delta)$ and have finite projective dimensions, the extension algebra $\Gamma = \text{Ext}_A^*(\Delta, \Delta)$ of standard modules is a graded finite-dimensional algebra, and provides us a lot information on the structures of indecomposable objects in $\mathcal{F}(\Delta)$. The structure of Γ has been considered in [2, 3, 7, 11, 17, 20, 21, 22, 23].

We describe some preliminary results in [13], where \leq is supposed to be a partial order.

Theorem 4.1. *If A is standardly stratified for a poset (Λ, \leq) , then the associated category \mathcal{E} of Γ is a directed category with respect to \leq and is standardly stratified for \leq^{op} . Moreover, \mathcal{E} is standardly stratified for \leq if and only if for all $\lambda, \mu \in \Lambda$ and $s \geq 0$, $\text{Ext}_A^s(\Delta_\lambda, \Delta_\mu)$ is a projective $\text{End}_A(\Delta_\mu)$ -module.*

Proof. This is Theorem 1.4 in [13]. □

We introduce some notations. Let Λ_1 be the subset of all minimal elements in Λ , Λ_2 be the subset of all minimal elements in $\Lambda \setminus \Lambda_1$, and so on. Then $\Lambda = \sqcup_{i \geq 1} \Lambda_i$. With this partition, we can introduce a *height function* $h : \Lambda \rightarrow \mathbb{N}$ in the following way: for $\lambda \in \Lambda_i \subseteq \Lambda$, $i \geq 1$, we define $h(\lambda) = i$. For each $M \in \mathcal{F}(\Delta)$, we define $\text{supp}(M)$ to be the set of elements $\lambda \in \Lambda$ such that $[M : \Delta_\lambda] \neq 0$. For example, $\text{supp}(\Delta_\lambda) = \{\lambda\}$. We also define $\min(M) = \min(\{h(\lambda) \mid \lambda \in \text{supp}(M)\})$. We say M is *generated in height i* if every simple summand of $M/\text{rad } M$ is isomorphic to some S_λ with $h(\lambda) = i$. For example, the standard module Δ_λ is generated in height $h(\lambda)$.

The following definition is an analogue of *Koszul modules* of graded algebras.

Definition 4.2. An A -module $M \in \mathcal{F}(\Theta)$ is said to be linearly filtered if there is some $i \in \mathbb{N}$ such that $\Omega_{\Theta}^s(M)$ is generated in height $i + s$ for $s \geq 0$. Equivalently, $M \in \mathcal{F}(\Theta)$ is linearly filtered if and only if it is generated in height i and has a projective resolution

$$0 \longrightarrow Q^l \longrightarrow Q^{l-1} \longrightarrow \dots \longrightarrow Q^{i+1} \longrightarrow Q^i \longrightarrow M \longrightarrow 0$$

such that each Q^s is generated in height s , $i \leq s \leq l$.

With this terminology, we have:

Theorem 4.3. Suppose that $\Delta \cong \Gamma_0$ as a Γ_0 -module and Δ_{λ} is linearly filtered for each $\lambda \in \Lambda$. Then:

- (1) Γ is a projective Γ_0 -module.
- (2) If $M \in \mathcal{F}(\Delta)$ is linearly filtered, then $\text{Ext}_A^*(M, \Theta)$ is a generalized Koszul Γ -module.
- (3) In particular, Γ is a generalized Koszul algebra.

Proof. The last two statements come from Theorem 2.12 in [13] by letting $\Theta = \Delta$ and $Q = {}_A A$. Thus we only need to show the first statement. But by (2.2) in page 14 of [13], we know that $\Gamma_s = \bigoplus_{\lambda \in \Lambda} \text{Ext}_A^s(\Delta_{\lambda}, \Delta)$ is a projective Γ_0 -module for every $s \geq 1$. \square

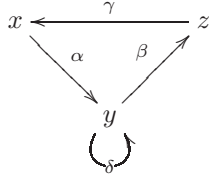
We remind the reader that although Γ is a generalized Koszul algebra, the Koszul duality in general does not hold since Γ_0 might not satisfy the splitting condition (S). See Example 2.14 in [13].

As before, let $\mathfrak{r} = \text{rad } \Gamma_0$ and $\bar{\Gamma} = \Gamma / \Gamma \mathfrak{r} \Gamma$. Then we have an immediate corollary.

Corollary 4.4. Suppose that $\Delta \cong \Gamma_0$ as a Γ_0 -module and Δ_{λ} is linearly filtered for each $\lambda \in \Lambda$. Then the quotient algebra $\bar{\Gamma}$ is a classical Koszul algebra.

Proof. This follows from Theorems 3.12 and 4.3. \square

Example 4.5. Let A be the path algebra of the following quiver with relations $\delta^2 = \delta\alpha = \beta\delta = \beta\alpha = \gamma\beta = 0$. Let $x > z > y$.



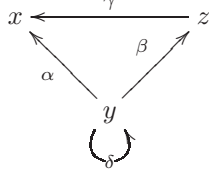
Indecomposable projective modules and standard modules of A are described below:

$$P_x = \begin{smallmatrix} x \\ y \end{smallmatrix} \quad P_y = \begin{smallmatrix} y \\ y \end{smallmatrix} \quad P_z = \begin{smallmatrix} z \\ x \\ y \end{smallmatrix}$$

$$\Delta_x = P_x = \begin{smallmatrix} x \\ y \end{smallmatrix} \quad \Delta_y = \begin{smallmatrix} y \\ y \end{smallmatrix} \quad \Delta_z = z.$$

Clearly, A is standardly stratified. Moreover, all standard modules have projective dimension 1 and are linearly filtered. By direct computation we check that $\Delta \cong \text{End}_A(\Delta)$ as $\Gamma_0 = \text{End}_A(\Delta)$ -modules.

Now we compute the extension algebra Γ : $\Gamma_s = 0$ for $s \geq 2$; $\text{Ext}_A^1(\Delta_x, \Delta) = 0$; $\text{Ext}_A^1(\Delta_y, \Delta) \cong \text{End}_A(\Delta_z)$; and $\text{Ext}_A^1(\Delta_z, \Delta) \cong \text{End}_A(\Delta_x)$. Therefore, we find Γ is the path algebra of the following quiver with relations $\delta^2 = \beta\delta = \alpha\delta = \gamma\beta = 0$.



We remind the reader that α is in the degree 0 part of Γ . Indeed, $\Gamma_0 = \langle 1_x, 1_y, 1_z, \delta, \alpha \rangle$ and $\Gamma_1 = \langle \beta, \gamma \rangle$. In this case Γ_0 does not satisfy the splitting condition (S) since we can find the following non-splitting exact sequence:

$$0 \rightarrow x \rightarrow \begin{matrix} y \\ y \end{matrix} \rightarrow \begin{matrix} y \\ x \end{matrix} \rightarrow y \rightarrow 0.$$

Since $\mathfrak{r} = \text{rad } \Gamma_0 = \langle \delta, \alpha \rangle = \Gamma \mathfrak{r} \Gamma$, the quotient algebra Γ is the path algebra of the following quiver with relation $\gamma\beta = 0$, which is clearly a classical Koszul algebra.

$$x \xleftarrow{\gamma} z \xleftarrow{\beta} y.$$

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